Noise properties of propagation through slow- and fast-light media

2010 J. Opt. 12 104007

(http://iopscience.iop.org/2040-8986/12/10/104007)

IOP PUBLISHING JOURNAL OF OPTICS
J. Opt. **12** (2010) 104007 (4pp) doi:10.1088/2040-8978/12/10/104007

Noise properties of propagation through slow- and fast-light media

Robert W Boyd[1,2], **Zhimin Shi**[1] **and Peter W Milonni**[2]

[1] The Institute of Optics, University of Rochester, Rochester, NY 14627, USA
[2] Department of Physics and Astronomy, University of Rochester, NY 14627, USA

Received 28 January 2010, accepted for publication 22 March 2010
Published 24 September 2010
Online at stacks.iop.org/JOpt/12/104007

Abstract
We consider the fundamental noise properties of propagation through slow- and fast-light optical media based on gain or loss processes. For purely quantum mechanical reasons, any gain or loss process will add noise to a transmitted light field. We derive a relation between the noise figure describing the decreased signal-to-noise ratio of the transmitted laser pulse and the fractional delay or advancement of the pulse. We apply these results explicitly to the situation of operation on the line center of a gain or loss line. We find that for an ideal gain medium the noise figure never exceeds a factor of two. For a loss medium, there is no limit as to how large the noise figure can become. The increased noise in this case is the result of the random loss of photons from the optical field.

Keywords: slow light, quantum noise

1. Introduction

In this paper we address the following question. How much noise, if any, is added to a beam of light as it propagates through a medium with an extreme value of the group velocity?

The motivation for this study is as follows. It is now well established that it is possible to find situations in which the group velocity of light v_g is significantly different from the vacuum speed of light c [1, 2]. In particular, one speaks of slow light for $v_g \ll c$ and fast light for $v_g > c$ or v_g negative. The special case of v_g negative is sometimes called backwards light. The most extreme values of the group index often occur when the signal wavelength is at or near a strong gain or absorption resonance of a material system. However, it is also well known that a beam of light becomes noisier upon propagation through a medium with gain or loss [3]. Thus one might expect light beams to become noisier upon propagation through slow- or fast-light media. In fact, recent studies have quantified the noise characteristics of slow light based on the process of electromagnetically induced transparency [4, 5].

In this paper, we examine the relation between the induced delay or advancement experienced by a light field in passing through a slow- or fast-light medium and the amount of noise that is acquired by such a field. We are primarily interested in determining any fundamental limits to the minimum amount of noise that must be added to the light field, and in determining how to minimize this noise. For this reason, we model the noise properties by means of Caves's quantum theory of the amplifier [3]. This model describes the minimum amount of noise that must, in accordance with the laws of quantum mechanics, be added to a light field. Of course, any physical amplifier can introduce additional noise over and above this minimum amount. This additional noise is often referred to as technical noise, and can result from thermal fluctuations or from interactions with other degrees of freedom of the material medium, such as acoustic phonons [6]. More general theoretical discussions of the noise properties of optical amplifiers that include the contribution of these additional noise sources have been presented previously [7–10].

Let us begin by considering the relationship between noise and delay/advancement from an abstract perspective. As is well known, the group velocity can be represented as $v_g = c/n_g$, where the group index is given by $n_g = n' + \omega\, dn'/d\omega$, where n' is the real part of the complex refractive index $n(\omega)$ and ω is an optical frequency. Throughout this work, we define the group index to be a purely real quantity. It is known from the Kramers–Kronig relations that a frequency dependence of the real part of the refractive index n' implies that the imaginary part of the refractive index n'' must be nonvanishing at *some* frequency. Thus gain or loss at some frequency is an intrinsic feature of dispersion of the real part of the refractive index. However, this conclusion does not require there to be gain or loss at the frequency of the signal wave whose velocity is to be modified. In fact, much recent work on slow and fast light has been motivated toward finding situations in which the gain or loss is minimized at the frequency at which the magnitude of the group index is maximized. One example of such an approach is to use electromagnetically induced transparency to eliminate loss at the frequency of high dispersion of the refractive index [12]. Another example is to work in the nearly transparent region between two gain [13, 14] or loss [15] lines. However, gain or loss can never be eliminated entirely, and some noise will thereby be introduced. Our procedure in this paper will therefore be to first develop some general results pertaining to the noise and dispersive properties of a material system. We will then apply these results to specific cases of interest. We note that the noise properties of fast-light propagation have been treated earlier [16], but from a somewhat different point of view. In section 5 of the present paper we will compare our results to those of this earlier work.

2. Quantum noise properties of an ideal optical amplifier

Let us begin by recalling why all amplifiers and attenuators must add noise to a beam of light. The theory of these effects is well established, but for completeness we present a brief review of the theoretical description of these effects. We begin by considering the case of an optical amplifier [3, 17], as illustrated in the upper part of figure 1. One might think that one could model such a device by assuming that

$$\hat{b} = g\hat{a}, \tag{1}$$

where $\hat{a}$ and $\hat{b}$ are the standard field operators for the input and output fields, respectively, and where g is the amplitude gain of the amplifier. However, a moment's thought reveals that this assumption cannot be correct, because both $\hat{b}$ and $\hat{a}$ are bosonic field operators and thus must satisfy the standard commutation relations

$$[\hat{a}, \hat{a}^\dagger] = 1 \qquad \text{and} \qquad [\hat{b}, \hat{b}^\dagger] = 1. \tag{2}$$

Equations (1) and (2) are clearly incompatible, because equation (1) implies that $[\hat{b}, \hat{b}^\dagger]$ must be $|g|^2$ times larger than $[\hat{b}, \hat{b}^\dagger]$, in contradiction to equation (2). We can find an amplifier relation that is compatible with equation (2) by assuming that

$$\hat{b} = g\hat{a} + \hat{L}^\dagger, \tag{3}$$

where $\hat{L}^\dagger$ represents some noise source operator. For the case of amplification by means of an inverted atomic system, this noise source can be identified with spontaneous emission. However, we will keep the physical nature of the noise source unspecified for our present purposes. We assume that this noise source is uncorrelated with the input field, which implies that $[\hat{a}, \hat{L}^\dagger] = [\hat{a}, \hat{L}] = 0$. If we also assume that the strength of this noise source is such that $\hat{L} = (|g|^2 - 1)^{1/2}\,\hat{c}$, where $\hat{c}$ is another photon field operator obeying $[\hat{c}, \hat{c}^\dagger] = 1$, we find that the photon commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{b}, \hat{b}^\dagger] = 1$ are both satisfied. Throughout this work we assume that $\langle \hat{n}_c \rangle \equiv \langle \hat{c}^\dagger \hat{c} \rangle = 0$, implying that the noise source is in its ground state and thus contributes the smallest possible amount of noise.

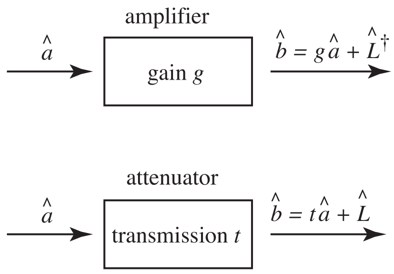

Figure 1. Schematic illustration of an ideal amplifier (upper) and an ideal attenuator (lower). $\hat{L}$ and $\hat{L}^\dagger$ represent Langevin noise sources.

Let us now examine the consequences of this noise source. We recall that the photon number operator for the input mode is $\hat{n}_a = \hat{a}^\dagger\hat{a}$ and that the photon number operator of the output mode is $\hat{n}_b = \hat{b}^\dagger\hat{b}$, which through use of equation (3) becomes

$$\hat{n}_b = |g|^2\hat{a}^\dagger\hat{a} + g^*\hat{a}^\dagger\hat{L}^\dagger + g\hat{a}\hat{L} + \hat{L}^\dagger\hat{L}. \tag{4}$$

By taking expectation values of each side of this equation, we find that

$$\langle \hat{n}_b \rangle = G\langle \hat{n}_a \rangle + G - 1, \tag{5}$$

where we have introduced the intensity gain parameter $G = |g|^2$. We see that, in addition to the input beam being amplified by a factor of G, $(G-1)$ noise photons are added to the output beam. Let us next consider the intensity fluctuations in the output beam. We find by a similar calculation that the variance of the photon number in the output beam is given by

$$\langle \Delta\hat{n}_b^2 \rangle = G^2\langle \Delta\hat{n}_a^2 \rangle + G(G-1)\,(\langle \hat{n}_a \rangle + 1). \tag{6}$$

Here the first term represents the amplification of fluctuations present in the input beam and the second term represents noise added to the beam by the amplifier.

It is useful to characterize the noise properties of an amplifier by means of its noise figure, which we define as the square of the ratio of signal-to-noise ratios of the input and output fields. (The square is included to render our definition consistent with the historical definition; see, for instance, [11].) Explicitly the noise figure is defined by

$$\mathcal{F} = \frac{(S/N)_a^2}{(S/N)_b^2}, \tag{7}$$

where

$$(S/N)_a^2 = \frac{\langle \hat{n}_a \rangle^2}{\langle \Delta\hat{n}_a^2 \rangle} \tag{8}$$

and

$$(S/N)_b^2 = \frac{G^2\langle \hat{n}_a \rangle^2}{\langle \Delta\hat{n}_b^2 \rangle}. \tag{9}$$

Note that in the latter expression we take the signal to be the amplified input field, not the total output field which includes the 'noise' contribution $G - 1$. By introducing expression (6) for $\langle \Delta\hat{n}_b^2 \rangle$, we find that

$$\mathcal{F} = 1 + \left(1 - \frac{1}{G}\right)\frac{\langle \hat{n}_a \rangle + 1}{\langle \Delta\hat{n}_a^2 \rangle}. \tag{10}$$

We now consider the important special case of an input beam that is strong in the sense that $\langle \hat{n}_a \rangle \gg 1$. We also assume that the fluctuations in the input beam possess Poissonian statistics in that $\langle \Delta \hat{n}_a^2 \rangle = \langle \hat{n}_a \rangle$. We then find that

$$\mathcal{F} \approx 2 - \frac{1}{G}. \tag{11}$$

It is interesting to examine how this noise figure varies with the amplifier gain parameter G. For $G = 1$ (no gain), the noise figure is equal to unity, showing that there is no change in the signal-to-noise ratio. In the limit of high gain, $G \gg 1$, the noise figure takes the value 2 under the assumptions made. We thus see that, under these conditions, the noise figure of an ideal amplifier lies always in the range from 1 to 2. We shall see next that an optical attenuator behaves very differently.

3. Quantum noise properties of optical attenuators

It is straightforward to repeat the calculation just presented for the case of an optical attenuator. One finds that, in order for the output field to possess standard commutation relations, the transmission properties of the attenuator must be described by

$$\hat{b} = t\hat{a} + \hat{L}, \tag{12}$$

where t is the amplitude transmission of the attenuator and where we now require that $\hat{L} = (1 - |t|^2)^{1/2}\hat{c}$ with $[\hat{c}, \hat{c}^\dagger] = 1$ and where we again assume that $\langle \hat{n}_c \rangle \equiv \langle \hat{c}^\dagger \hat{c} \rangle = 0$. We can then calculate the noise figure of the attenuator and find it to be given by

$$\mathcal{F} = 1 + \left(\frac{1}{T} - 1\right) \frac{\langle \hat{n}_a \rangle}{\langle \Delta \hat{n}_a^2 \rangle}, \tag{13}$$

where $T = |t|^2$ is the intensity transmission. For the special case in which the fluctuations in the input field obey Poisson statistics, the noise figure reduces to

$$\mathcal{F} = \frac{1}{T}. \tag{14}$$

Note that, unlike for the case of the optical amplifier, the noise figure of an optical attenuator can becomes arbitrarily large, which occurs in the limit of small transmission. This difference, of course, will influence the noise characteristics of slow- and fast-light media. These results are summarized in figure 2, which shows how the noise figure of an ideal attenuator or amplifier depends on the transmission through the device.

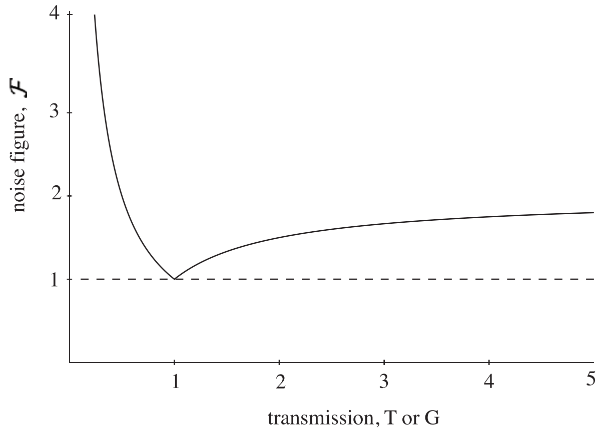

Figure 2. Noise figure of an ideal optical amplifier or attenuator as predicted by equations (11) and (14) plotted against the transmission through the device. These predictions assume that the fluctuations of the input beam obey Poissonian statistics and that, for the case of the amplifier ($G > 1$), the mean photon number of the input beam satisfies $\langle \hat{n}_a \rangle \gg 1$.

4. Noise properties of specific slow-/fast-light systems

Let us now consider a specific slow-light situation. We consider an optical medium that possesses a single gain resonance of line-center gain coefficient g_0 and linewidth γ centered at frequency ω_0. We can describe such a medium in terms of a complex refractive index given by

$$n(\omega) = 1 + \frac{g_0 \lambda}{4\pi} \frac{\gamma}{\omega - \omega_0 + \mathrm{i}\gamma}. \tag{15}$$

By means of a simple calculation, we find that the group index $n_\mathrm{g} = n' + \omega\,(\mathrm{d}n'/\mathrm{d}\omega)$ at line center for a medium of this sort is given by

$$n_\mathrm{g} = 1 + \frac{\omega}{\gamma} \frac{g_0 \lambda}{4\pi} = 1 + \frac{g_0 c}{2\gamma}. \tag{16}$$

The induced time delay on transit through this medium, relative to vacuum propagation, is thus given by

$$\tau_\mathrm{g} = \frac{(n_\mathrm{g} - 1)L}{c} = \frac{g_0 L}{2\gamma}. \tag{17}$$

In writing this result in this form, we have implicitly assumed that the light pulse is spectrally sufficiently narrow that pulse distortion due to dispersion of the group velocity is small. In this case the time delay is reliably estimated by means of the group velocity.

We next determine the amount of noise added to this light field. Through use of equation (11) we find that

$$\mathcal{F} = 2 - \frac{1}{G} = 2 - \frac{1}{\exp(g_0 L)} \tag{18}$$

or through use of equation (17) that

$$\mathcal{F} = 2 - \frac{1}{\exp(2\tau_\mathrm{g}\gamma)}. \tag{19}$$

From this result we see that the noise in the output beam increases with increasing delay, but that the noise figure never exceeds a factor of two.

We can perform an analogous calculation for the case of a fast-light medium described by an isolated absorption line [18]. For this situation the expressions for the refractive index and group index are given by expressions of the same form as equations (15) and (16) but with g_0 replaced by $-\alpha_0$. The pulse advancement is thus given by

$$\tau_\mathrm{a} = -\frac{(n_\mathrm{g} - 1)L}{c} = \frac{\alpha_0 L}{2\gamma}. \tag{20}$$

The noise properties in this case are described by equation (14). We thus find that

$$\mathcal{F} = \frac{1}{T} = \frac{1}{\exp(-\alpha_0 L)} \tag{21}$$

or through use of equation (20) that

$$\mathcal{F} = \frac{1}{\exp(-2\tau_a \gamma)}. \tag{22}$$

This equation shows that formally the noise figure can become arbitrarily large for large values of the time advancement. There seems to be no fundamental limit on how large the fractional advancement (advancement measure in units of pulse width) can become [19]. However, there is strong empirical evidence that the time advancement can never exceed several pulse widths [20–22], and thus that the parameter $\tau_a\gamma$ cannot become much larger than unity. We thus conclude that the noise figure cannot become larger than approximately 10.

5. Discussion and summary

We have noted that slow-light methods often employ media that produce some level of gain or loss at the frequency of the signal wave, and that as a consequence some amount of noise is always added to the signal wave. We have quantified this effect by determining the noise figure of a slow-light medium, which is defined to be the factor by which the square of the signal-to-noise ratio of a signal pulse is decreased as the pulse propagates through the medium. For the case of a slow-light medium based upon an ideal gain resonance, the noise figure is never larger than a factor of two, which shows that noise resulting from the slow-light process itself is not expected to play a major role under these conditions. We have treated the case of line-center operation, where the influence of gain might be expected to be largest, but we find that the signal-to-noise ratio of the transmitted pulse is not significantly degraded. On the other hand, for the case of fast light based on an absorption resonance, there is no formal limit as to how large the noise figure can become. However, once the empirical constraint on the maximum observable time advancement is included in the model, we conclude that the noise figure can never exceed a factor of ten. The noise in this case can be identified as arising solely from the random loss of photons from the signal field.

A related treatment of the noise aspects of fast-light propagation has been presented by Kuzmich *et al* [16]. These authors conclude that quantum noise does play a significant role in imposing causality in fast-light propagation. In particular, these authors define a signal velocity in terms of what one could actually measure under realistic laboratory conditions, and conclude that quantum noise prevents the signal velocity so defined from becoming greater than c. This conclusion seems to be at odds with the conclusion reached by the present work. We believe that the differing conclusion results from the fact that the present treatment models the entire pulse as a single field mode and asks how much noise is added to that mode. This sort of analysis might be most relevant for analyzing the noise properties of a telecommunications system. In contrast, Kuzmich *et al* [16] were interested in the amount of noise added to the very leading edge of an optical pulse. It seems likely that the differences in the way the question has been formulated leads to the somewhat different conclusions of these two treatments.

Acknowledgments

RWB and ZS gratefully acknowledge support of the US National Science Foundation and from the US DARPA Slow Light Program.

References

[1] Milonni P W 2005 *Fast Light, Slow Light, and Left-Handed Light* (Bristol: Institute of Physics Publishing)
[2] Boyd R W and Gauthier D J 2009 *Science* **326** 1074
[3] Caves C M 1982 *Phys. Rev.* D **26** 1817
[4] Hsu M T L *et al* 2006 *Phys. Rev. Lett.* **97** 183601
[5] Hetet G *et al* 2008 *Opt. Express* **16** 7369
[6] Tkach R W and Chraplyvy A R 1989 *Opt. Quantum Electron.* **21** S105
[7] Yamamoto Y and Mukai T 1989 *Opt. Quantum Electron.* **21** S1
[8] Hinton K 1989 *Opt. Quantum Electron.* **21** S33
[9] Donati S and Giuliani G 1997 *IEEE J. Quantum Electron.* **33** 1481
[10] Yamamoto Y and Inoue K 2003 *J. Light. Technol.* **21** 2895
[11] Choi S-K, Vasilyev M and Kumar P 1999 *Phys. Rev. Lett.* **83** 1938
[12] Hau L V, Harris S E, Dutton Z and Behroozi C 1999 *Nature* **397** 594
[13] Steinberg A M and Chiao R Y 1994 *Phys. Rev.* A **49** 2071
[14] Wang L J, Kuzmich A and Dogariu A 2000 *Nature* **406** 277
[15] Camacho R M, Pack M V, Howell J C, Schweinsberg A and Boyd R W 2007 *Phys. Rev. Lett.* **98** 153601
[16] Kuzmich A, Dogariu A, Wang L J, Milonni P W and Chiao R Y 2001 *Phys. Rev. Lett.* **86** 3925
[17] Boyd R W, Agarwal G S, Davis W V, Gaeta A L, Nagasako E M and Kauranen M 1994 *Acta Phys. Pol.* A **86** 117
[18] Chu S and Wong S 1982 *Phys. Rev. Lett.* **48** 738
[19] Boyd R W, Gauthier D J, Gaeta A L and Willner A E 2005 *Phys. Rev.* A **71** 023801
Boyd R W, Gauthier D J, Gaeta A L and Willner A E 2005 *Phys. Rev.* A **72** 059903 (erratum)
[20] Segev B, Milonni P W, Babb J F and Chiao R Y 2000 *Phys. Rev.* A **62** 022114
[21] Khurgin J B 2006 *Opt. Lett.* **31** 948
[22] Boyd R W and Narum P 2007 *J. Mod. Opt.* **54** 2403